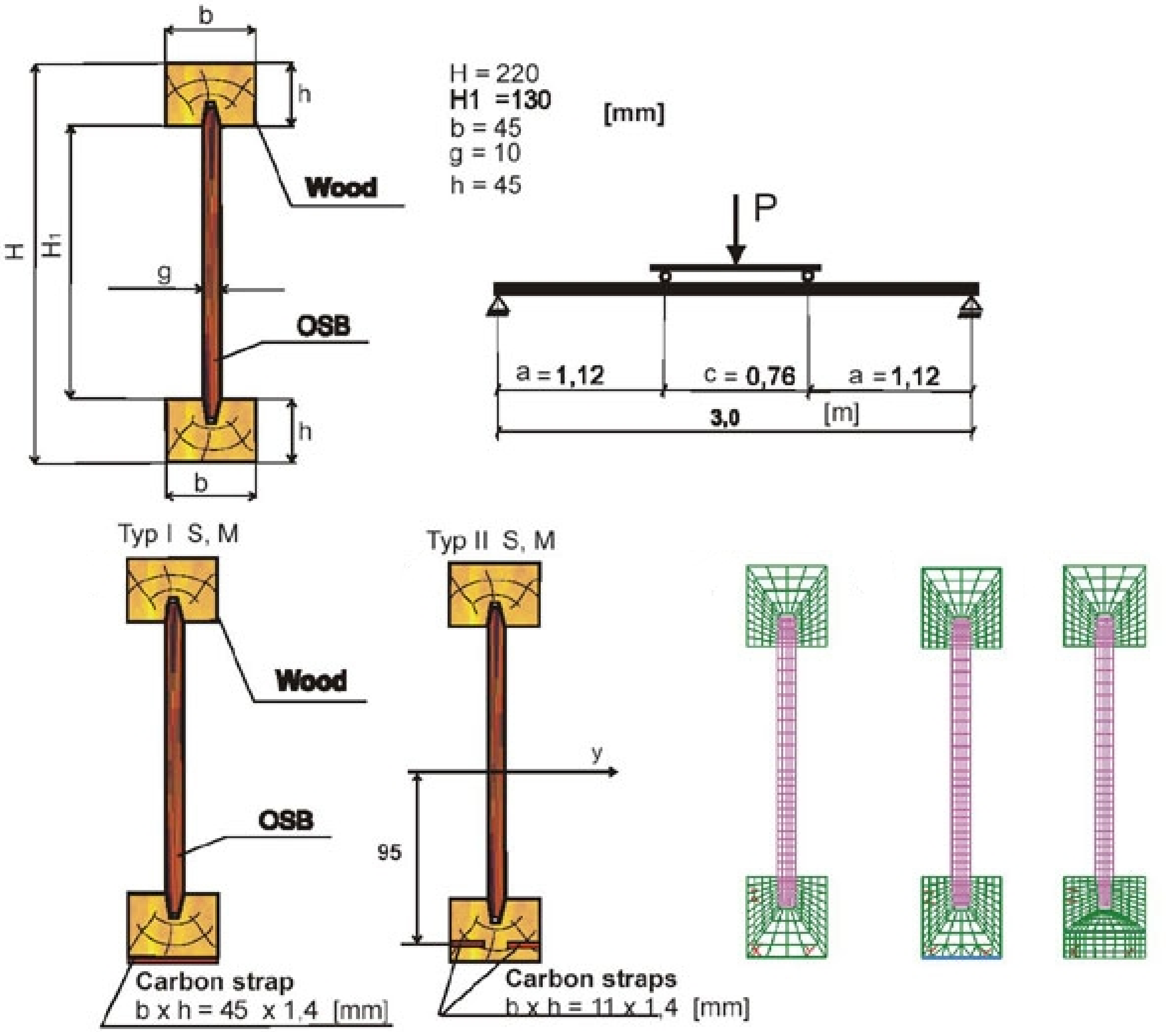
# COMPOSITE MOMENT OF INERTIA




Built-up Timber I-beams
Courtesy Electronic Journal of
Polish Agricultural Universities

# Moment of Inertia for Composite Areas

- Moments of inertia are additive if they reference the same axis. That is:

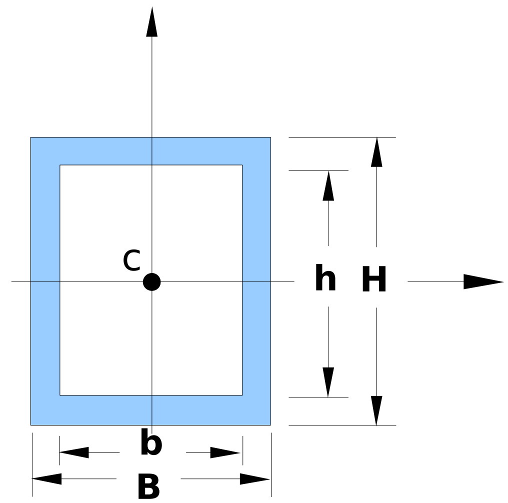
$$I_x = \sum_{i=1}^{n} I_x^{(i)} \qquad \text{and} \qquad I_y = \sum_{i=1}^{n} I_y^{(i)}$$

  We can use this to our advantage for determination of composite cross sections.

- For our discussion, a composite cross section is one comprised of mutiple simple geometric shapes. One of the simplest composite shapes is a round or rectangular tube.

# Moment of Inertia for Composite Areas

- Consider the square tube shown below.

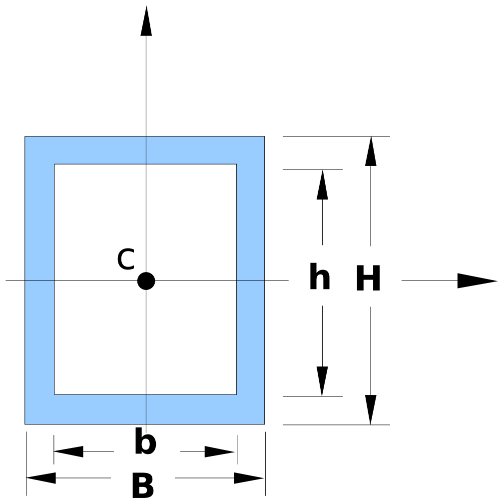



- From the appendix, we know the moment of inertia of a rectangle about each of its centroidal axis is:

$$I_x = \frac{bh^3}{12} \qquad I_y = \frac{hb^3}{12}$$

# Moment of Inertia for Composite Areas

- The square tube can be modeled as two concentric rectangles with a common x- and y-axis. This allows the moment of inertia of each shape to be added algebraically. Since the interior rectangle is a 'hole', treat this as a "negative area" and add a negative area and a negative moment of inertia.




$$I_x = \frac{BH^3}{12} - \frac{bh^3}{12}$$

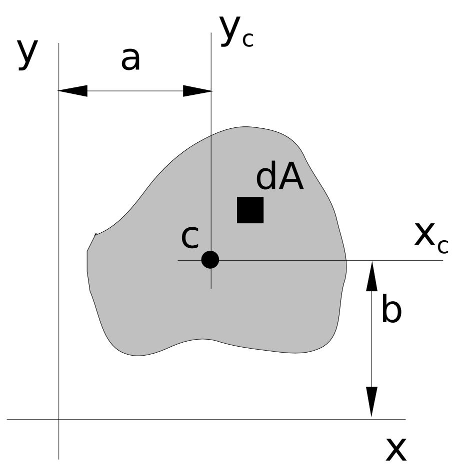
$$I_y = \frac{HB^3}{12} - \frac{hb^3}{12}$$

# Parallel Axis Theorem for Moment of Inertia

- Since moments of inertia can only be added if they reference the same axis, we must find a way to determine the moments of inertia of composite sections when this is not the case.
- An example of this is the concrete T-beam shown. Although it is a simple matter to determine the moment of inertia of each rectangular section that makes up the beam, they will not reference the same axis, thus cannot be added.
- However, if we found the moment of inertia of each section about some reference axis such as the centroidal axis of the composite, then we could add the individual moments of inertia.

# Parallel Axis Theorem for Moment of Inertia

- The parallel axis theorem is used to do just that.  Consider the following area with a known centroid.



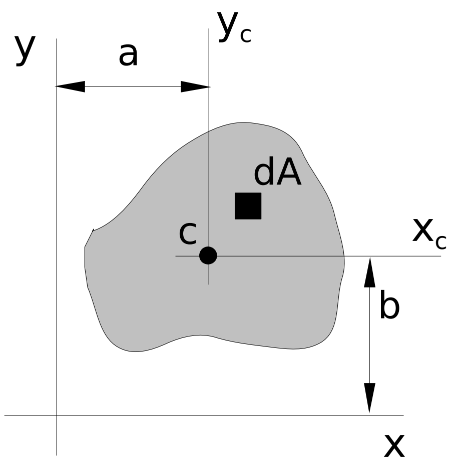

$$\text{Let} \quad \left\{ \begin{aligned} x &= x_c + a \\ y &= y_c + b \end{aligned} \right\}$$

$$\text{Then} \quad \left\{ \begin{aligned} \bar{x}_c &= \frac{\int x_c \, dA}{A} = 0 \\ \bar{y}_c &= \frac{\int y_c \, dA}{A} = 0 \end{aligned} \right\} \quad \text{location of centroid}$$

$$I_x = \int_A y^2 \, dA = \int_A (y_c + b)^2 \, dA = \underbrace{\int_A y_c^2 \, dA}_{I_{xc}} + \underbrace{2\mathrm{b} \int_A y_c \, dA}_{0} + \underbrace{b^2 \int_A dA}_{Area}$$

$$I_x = I_{xc} + b^2 \cdot A$$

# Parallel Axis Theorem for Moment of Inertia

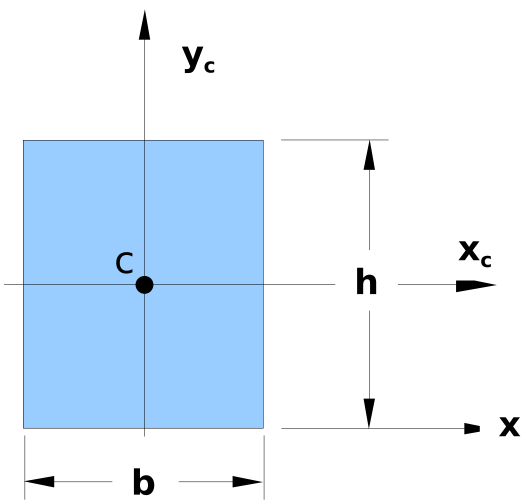



$$I_x = I_{xc} + b^2 \cdot A$$
$$I_y = I_{yc} + a^2 \cdot A$$

- Thus, the area moment of inertia with respect to any axis in its plane is equal to the moment of inertia with respect to the parallel centroidal axis plus the product of the area and the square of the distance between the two axis.

# Examples for Moment of Inertia

- Example 4:
  Given the moment of inertia of a rectangle about its centroidal axis, apply the parallel axis theorem to find the moment of inertia for a rectangle about its base.




$$I_{xc} = \frac{b \cdot h^3}{12}$$

$$distance = \frac{h}{2} \qquad area = b \cdot h$$

$$I_x = \frac{b \cdot h^3}{12} + \left(\frac{h}{2}\right)^2 \cdot (b \cdot h)$$

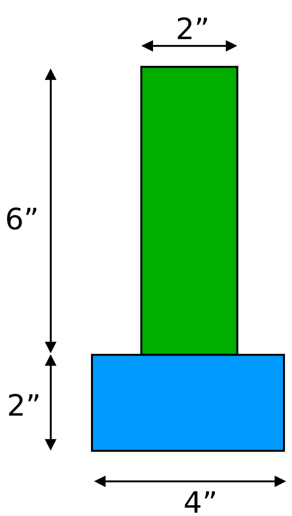
$$= \frac{b \cdot h^3}{12} + \frac{b \cdot h^3}{4}$$

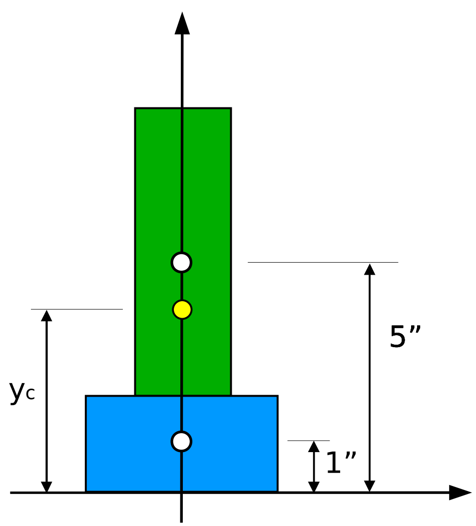
$$= \frac{b \cdot h^3}{12} + \frac{3 \cdot b \cdot h^3}{12} = \boxed{\frac{b \cdot h^3}{3}}$$

# Examples for Moment of Inertia

- Example 5:
  Find the centroidal moment of inertia for a T-shaped area.

1) First, locate the centroid of each rectangular area relative to a common base axis, then...
2) ...determine the location of the centroid of the composite.




$$x_c = 0 \qquad \text{(by symmetry)}$$

$$y_c = \frac{\Sigma\, y_i A_i}{\Sigma\, A_i} = \frac{(5\cdot(6\cdot 2)+1\cdot(2\cdot 4))}{(2\cdot 4+6\cdot 2)} = \frac{(60+8)}{20} = 3.4\ \text{in}$$

# Examples for Moment of Inertia

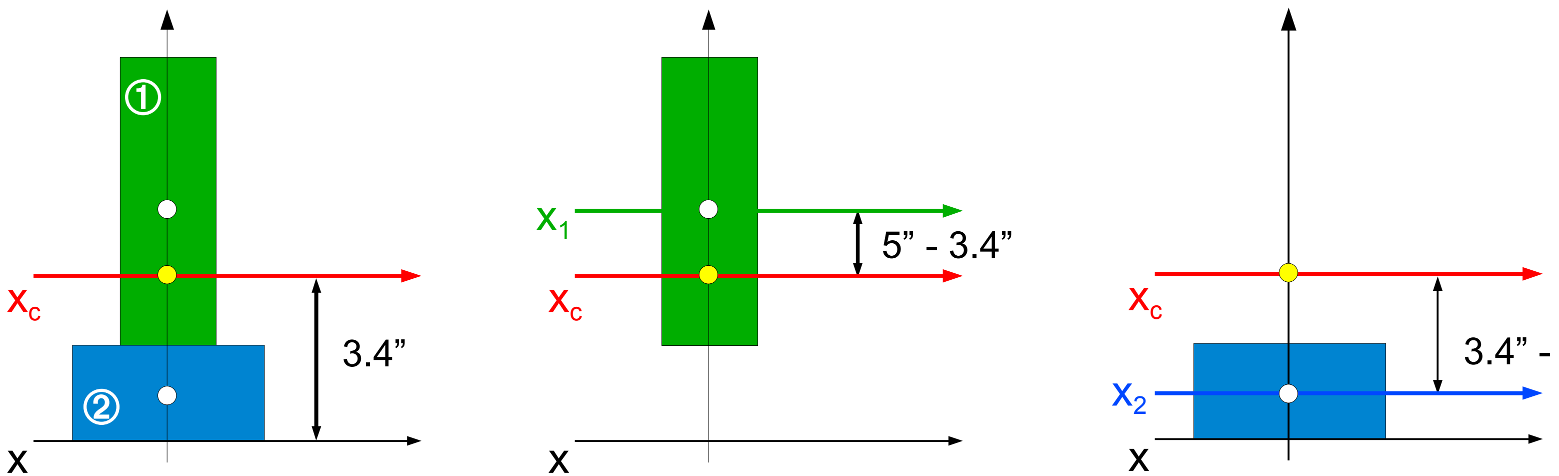
3) Find centroidal moment of inertia about the x-axis

$$I_{xc} = I_{x1} + I_{x2} \quad \text{where} \quad I_{xi} = \frac{b_i \cdot h_i^3}{12} + d_i^2 \cdot (b_i \cdot h_i)$$

$$I_{x1} = \frac{2 \cdot 6^3}{12} + (5 - 3.4)^2 \cdot 2 \cdot 6 = 66.72 \cdot \text{in}^4$$

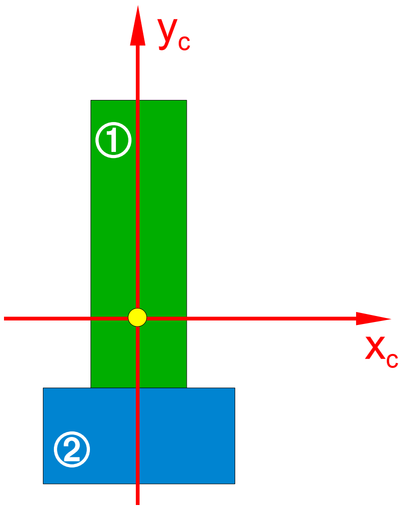
$$I_{x2} = \frac{4 \cdot 2^3}{12} + (3.4 - 1)^2 \cdot 2 \cdot 4 = 48.75 \cdot \text{in}^4$$

$$I_{xc} = I_{x1} + I_{x2} = 66.72 + 48.75 = 115.46 \text{ in}^4$$



# Examples for Moment of Inertia

4) Find the centroidal moment of inertia about the y-axis.
Since the centroidal y-axis for each shape and for the composite is coincident, the moments of inertia are additive.




$$I_{yc} = I_{y1} + I_{y2}$$

$$I_{yi} = \frac{h_i \cdot b_i^3}{12}$$

$$I_{yc} = \frac{6 \cdot 2^3}{12} + \frac{2 \cdot 4^3}{12} = 14.66 \cdot \text{in}^4$$

# Examples for Moment of Inertia

- As sections become more complex, it is often easier to perform the calculations by creating tables to find centroid and moment of inertia.

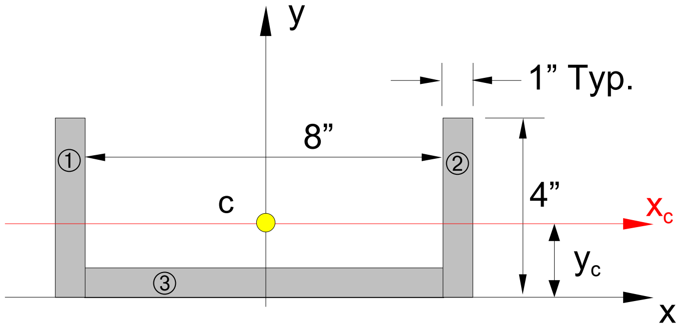
| Part | Dimensions | Area | x | y | xA | yA |
|---|---|---|---|---|---|---|
| 1 | | | | | | |
| 2 | | | | | | |
| 3 | | | | | | |
| Total | ---------- | | ---- | ---- | | |

| Part | Area | Ix | Iy | dy | dx | $d^2y(A)$ | $d^2x(A)$ | $Ix + d^2y(A)$ | $Iy + d^2x(A)$ |
|---|---|---|---|---|---|---|---|---|---|
| 1 | | | | | | | | | |
| 2 | | | | | | | | | |
| 3 | | | | | | | | | |
| Total | | ---- | ---- | ---- | ---- | ---- | ---- | | |

# Examples for Moment of Inertia

- Example 6:
  Determine the location of the centroid ('c') of the beam's cross section and the moment of inertia about the centroidal x-axis.

| Part | Dimensions | Area | y | yA |
|---|---|---|---|---|
| 1 | 4 x 1 | 4 | 2 | 8 |
| 2 | 4 x 1 | 4 | 2 | 8 |
| 3 | 1 x 8 | 8 | 0.5 | 4 |
| Total | ---------- | **16 in$^2$** | ---- | **20 in$^3$** |

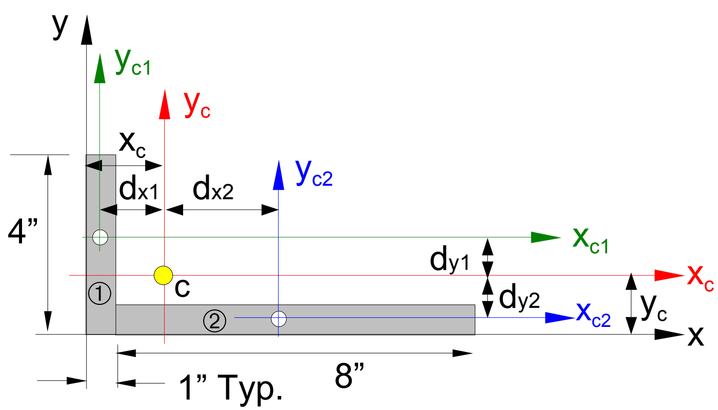



$$\bar{y}_c = \sum \frac{\bar{y} \cdot A}{A} = \frac{20}{16} = 1.25 \text{ in}$$

| Part | Area | $I_{ci}$ | $d_y$ | $d^2_y(A)$ | $I_{xci} = I_{ci} + d^2_y(A)$ |
|---|---|---|---|---|---|
| 1 | 4 | $1 \cdot (4)^3/12$ | 2 - 1.25 | 2.25 | 5.33 + 2.25 = 7.58 |
| 2 | 4 | $1 \cdot (4)^3/12$ | 2 - 1.25 | 2.25 | 5.33 + 2.25 = 7.58 |
| 3 | 8 | $8 \cdot (1)^3/12$ | 1.25 - 0.5 | 4.50 | 0.666 + 4.50 = 5.16 |
| Total | **16 in$^2$** | ---- | ---- | ---- | **20.32 in$^4$** |

# Examples for Moment of Inertia

- Example 7:
  Determine the location of the centroid 'c' the beam's cross section and the moment of inertia about both centroidal axis.




| Part | Dimensions | Area | x | y | xA | yA |
|---|---|---|---|---|---|---|
| 1 | 4 x 1 | 4 | 0.5 | 2 | 2 | 8 |
| 2 | 1 x 8 | 8 | 5 | 0.5 | 40 | 4 |
| Total | ---------- | **12** | ---- | ---- | **42** | **12** |

$$x_c = \sum \frac{x_i \cdot A_i}{A} = \frac{40}{12} = 3.33 \text{ in}$$

$$y_c = \sum \frac{y_i \cdot A_i}{A} = \frac{12}{12} = 1.00 \text{ in}$$

| Part | Area | Ix | $d_{y1}$ | $d^2_{y1}$ (A) | Ix + $d^2_{y1}$ (A) | Iy | $d_{x2}$ | $d^2_{x2}$ (A) | Iy + $d^2_{x2}$ (A) |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 4 | 5.333 | 2 - 1 | 4 | 9.33 | 0.333 | 3.5 - 0.5 | 36 | 36.33 |
| 2 | 8 | 0.666 | 1 - 0.5 | 2 | 2.67 | 42.66 | 5 - 3.5 | 18 | 60.66 |
| Total | **12** | ---- | ---- | ---- | **12** | ---- | ---- | -------- | **97** |