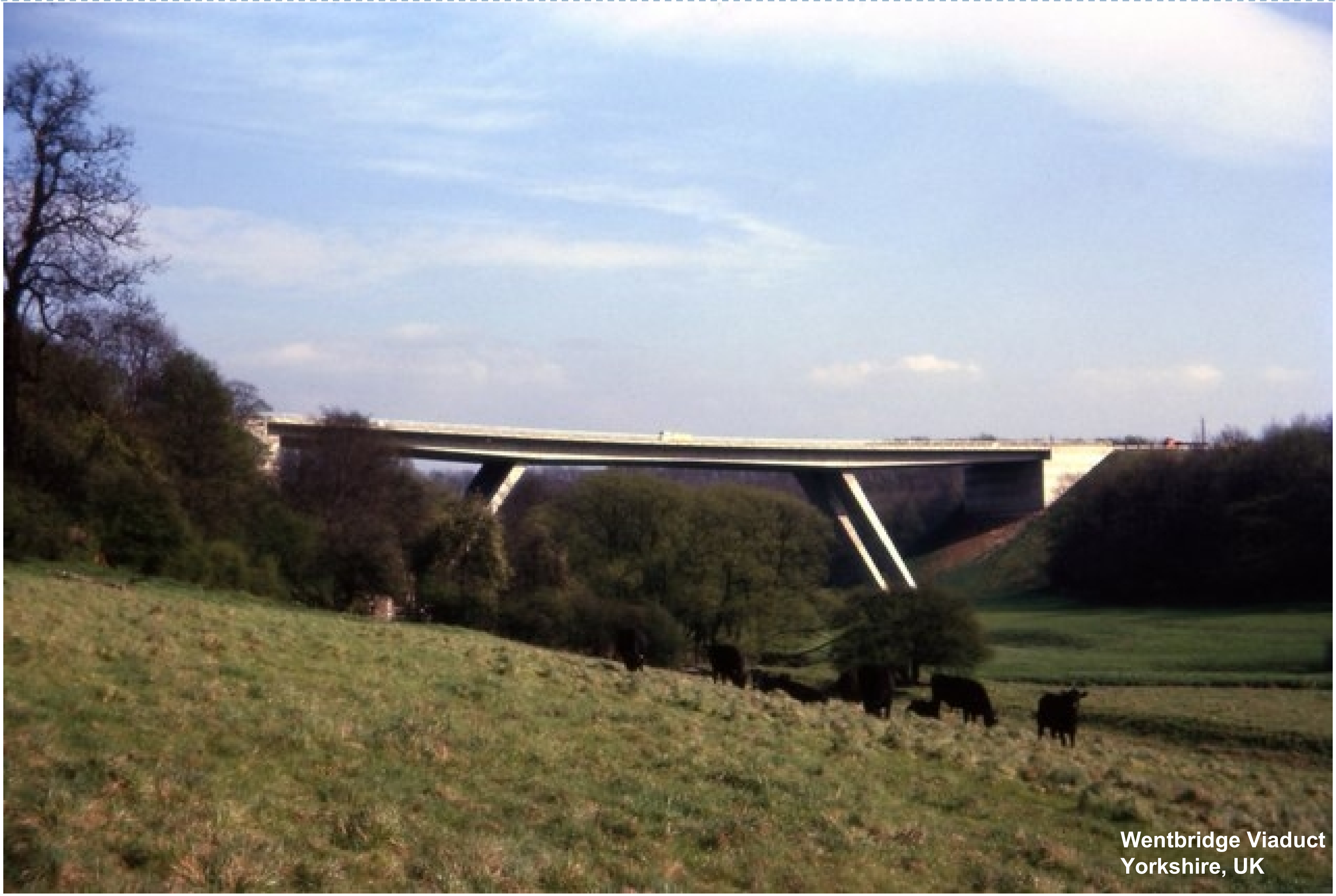
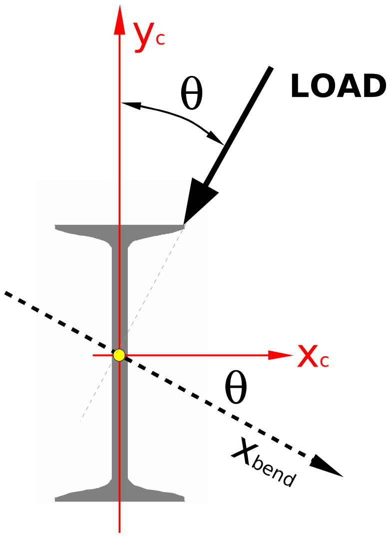
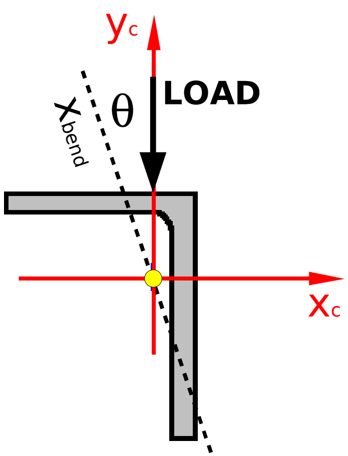
MOHR'S CIRCLE FOR MOMENT OF INERTIA



Asymmetrical Sections and Loads

- To this point, calculation of moment of inertia has been based upon the section being loaded symmetrically
- However, if load is applied at a different angle, the section will bend about axis x_{bend}
- Similarly, an asymmetrical section, such as L-shape, even when loaded perpendicular to its centroidal axis, will bend about a different axis x_{bend}
- We need to be able to find the rotation of and the moment of inertia about axis x_{bend}





Asymmetrical Sections and Loads

- Finding the moment of inertia for a section about some arbitrary axis x_{bend} is most easily done by constructing Mohr's circle
- To construct Mohr's circle for moment of inertia, we need to know three things:
  - Centroidal second moment of area about the x-axis
  - Centroidal second moment of area about the y-axis
  - Product second moment of area relative to the centroidal x-y axis
- We know how to find the first two. Let's develop the product second moment of area

Defining Product of Inertia

- The product second moment of area, hereafter refered to as the product of inertia, is mathematically defined as:

$$I_{xy} = \int_A (x \cdot y)\, dA$$

dA = elemental area
x = distance of dA from centroidal y-axis
y = distance of dA from centroidal x-axis

algebraically, this is expressed as:

$$I_{xy} = \sum_i \left(x \cdot y \cdot A_i \right)$$

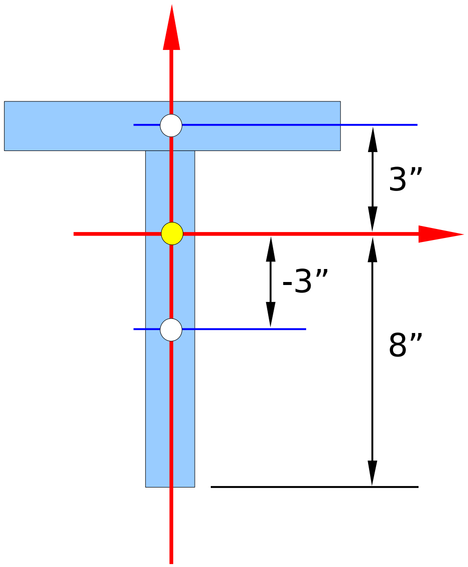
- One should note the distances defined by 'x' and 'y' may be either positive or negative, thus the product of inertia may be either positive or negative

Defining Product of Inertia

- This product of inertia will always be zero if a section has at least one axis of symmetry (i.e.: I-beams, C-channels, etc.)
- On the other hand, the product of inertia will be non-zero if the section is asymetrical (i.e.: L-section)
- The product of inertia can only be zero about the principle axis. The importance of this fact is that this is when moment of inertia is at its maximum and minimum values
- We will verify these ascertions within the following pages

Finding Product of Inertia

- Example 10
  Determine the moment of inertia and the product of inertia of a wooden T-beam section. Each leg is comprised of a 2 x 10. The centroid is 8” above the base. The centroidal moments of inertia and the product of inertia are determined using the table below

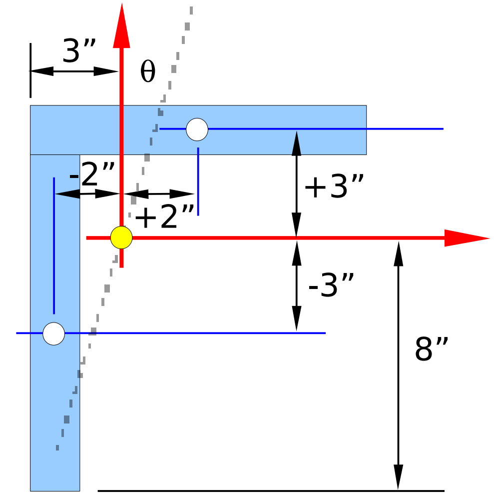


Part	Area	Ix	d_{y1}	d^2_{y1} (A)	Ix + d^2_{y1} (A)	Iy	d_{x2}	d^2_{x2} (A)	Iy + d^2_{x2} (A)	A (d_x)(d_y)
1	20	6.67	3	180	186.7	166.7	0	0	166.7	0
2	20	166.7	-3	180	346.7	6.67	0	0	6.67	0
Total	40	----	---	----	533	----	----	--------	173	0

Product of inertia = I_{xy} = A (d_x)(d_y) = 0

Finding Product of Inertia

- Example 11
  Determine the moment of inertia and the product of inertia of a wooden L-section. Each leg is comprised of a 2 x 10. The centroid is 8” above the base and 3” from the left edge. The centroidal moments of inertia and the product of inertia are determined using the table below



Part	Area	Ix	d_{y1}	d^2_{y1} (A)	Ix + d^2_{y1} (A)	Iy	d_{x2}	d^2_{x2} (A)	Iy + d^2_{x2} (A)	A (d_x)(d_y)
1	20	6.67	+3	180	186.7	166.7	+2	80	246.7	120
2	20	166.7	-3	180	346.7	6.67	-2	80	86.67	120
Total	40	----	---	----	533	----	----	--------	333	240

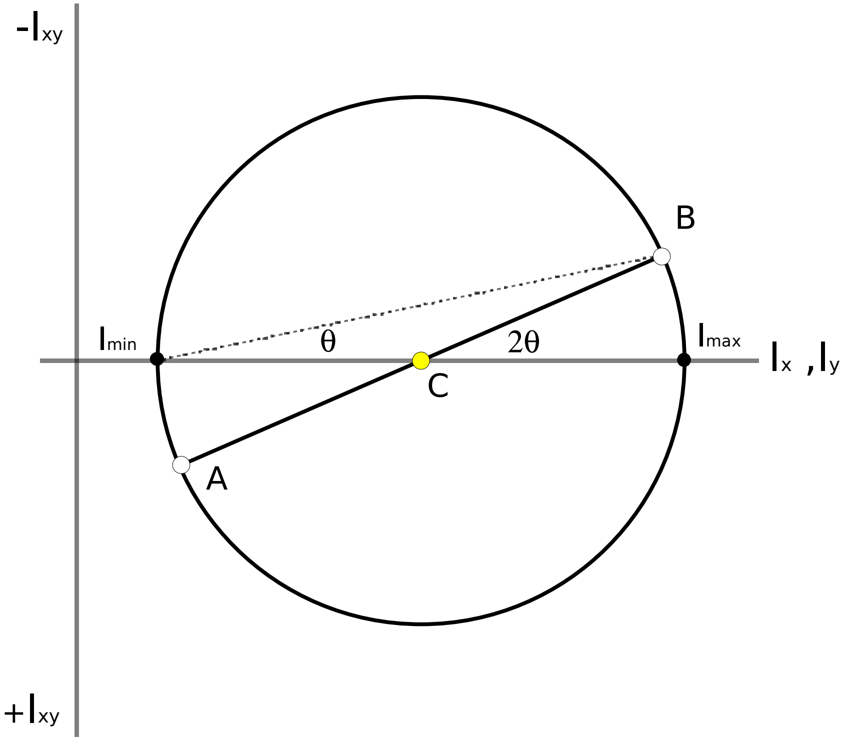
Product of inertia = $I_{xy} = A (d_x)(d_y) = 240\ in^4$

The Principal Axis

- The previous examples illustrate the product of inertia is zero for sections with one or two axis of symmetry while asymetrical sections have a non-zero product of inertia
- Now we need to locate the principal axis for the asymmetrical section by finding the angle θ as shown in Example 11
- This is where Mohr's circle and the equations for Mohr's circle become useful; it is a graphical means of finding the rotation of the principal axis

To Construct Mohr's Circle...

- ...construct an axis with $-I_{xy}$ in the postive y-direction and with I_x and I_y on the x-axis
- Plot points A and B. The x-y coordinates of point A are the minimum of I_x or I_y and $-I_{xy}$. Then point B has coordinates that are the maximum of I_y or I_x and $+I_{xy}$
- Draw the diameter and locate the center 'C'. The angle the diameter makes with the x-axis is twice the angle of rotation of the principal axis
- Draw the circle. The points where the circle intersect the x-axis define the maximum and minimum moments of inertia with a zero product of inertia



To Calculate Mohr's Circle

- Rather than draw Mohr's Circle, one can also calculate the rotation of the principle axis as well as the minimum and maximum values of moment of inertia

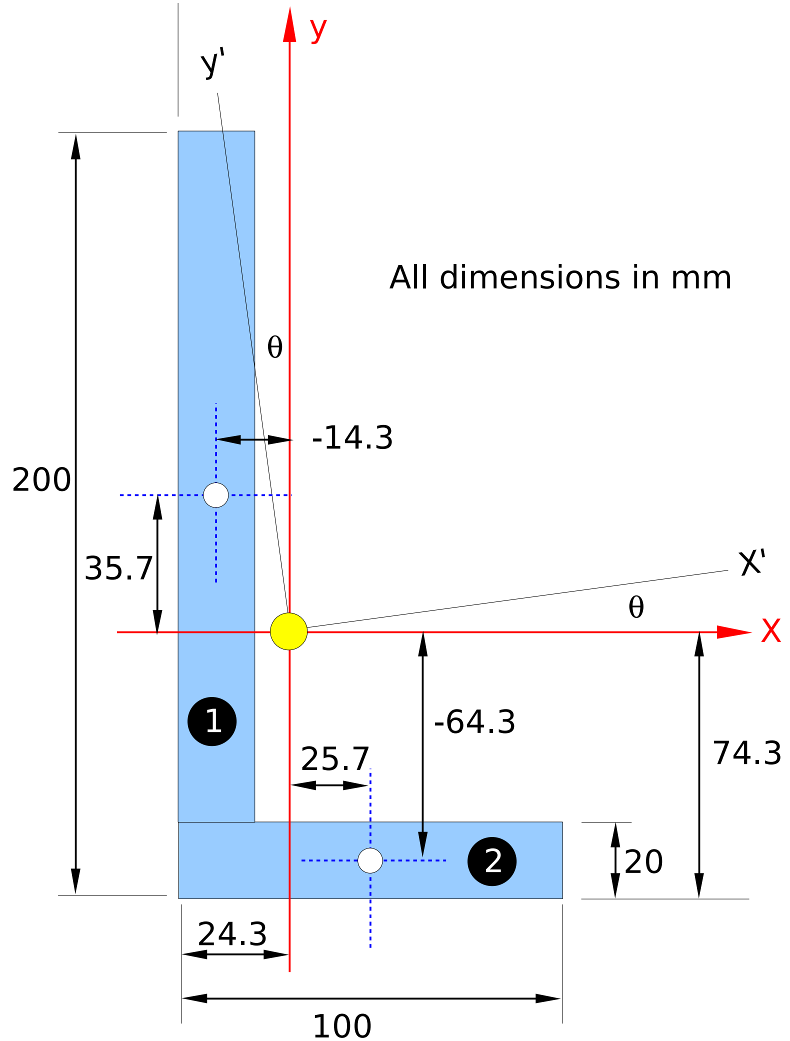
$$\tan 2\theta = \frac{-I_{xy}}{(I_x - I_y)/2}$$

$$I_{(x',y')} = \frac{I_x + I_y}{2} \pm \left[\left(\frac{I_x - I_y}{2}\right)^2 + I_{xy}^2\right]^{\frac{1}{2}}$$

- Note that $I_x + I_y = I_{x'} + I_{y'}$

Example of Mohr's Circle for Moment of Inertia

- A steel angle has dimensions of 200 mm x 100 mm x 20 mm. The centroid is at 74.3 mm from the bottom and 24.3 mm from the left face
  - Construct Mohr's circle for moment of inertia
  - Determine the rotation angle of the principle axis
  - Determine the maximum and minimum values of moment of inertia



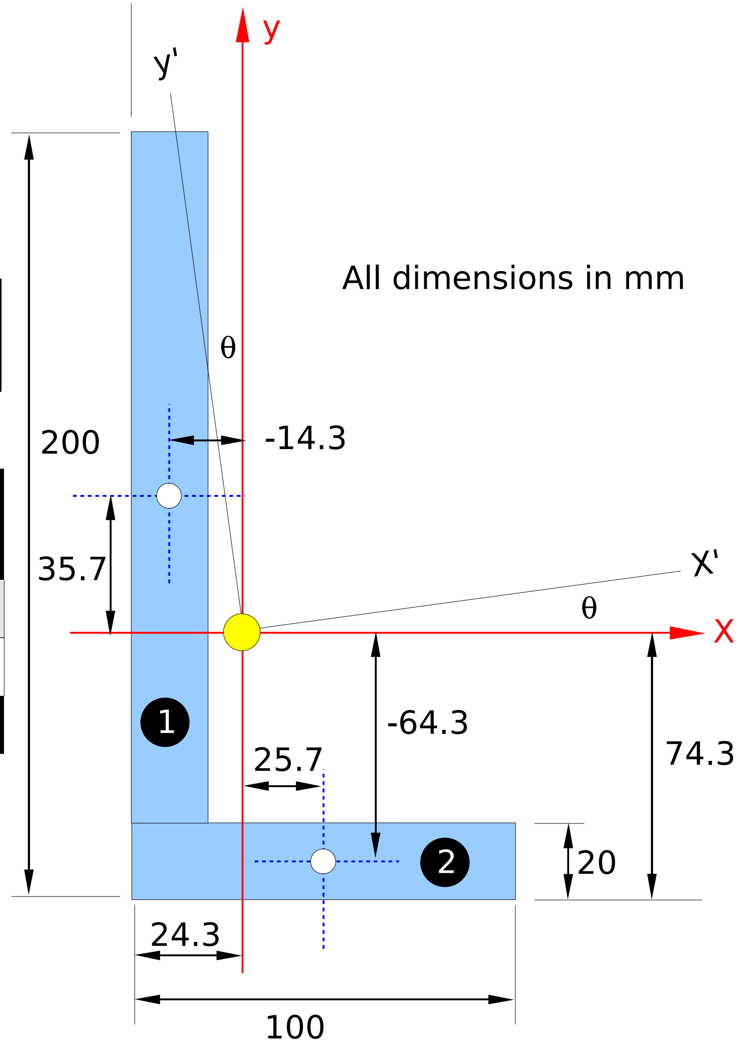
Example of Mohr's Circle for Moment of Inertia

- Use the table approach to help solve this problem.

Part	Area mm^2	I_x mm^4 (10^{-6})	d_y mm^4 (10^{-6})	d^2_y (A) mm^4 (10^{-6})	$I_x + d^2_y$ (A) mm^4 (10^{-6})
1	3600	9.72	35.7	4.59	14.31
2	2000	$0.066\overline{6}$	-64.3	8.27	8.33
Total	5600	----	---	----	22.64

Part	Area mm^2	I_y mm^4 (10^{-6})	d_x mm^4 (10^{-6})	d^2_x (A) mm^4 (10^{-6})	$I_y + d^2_x$ (A) mm^4 (10^{-6})
1	3600	0.120	-14.3	0.736	0.856
2	2000	$1.6\overline{6}$	25.7	1.32	2.99
Total	5600	----	----	--------	3.85

Part	Area mm^2	d_y mm	d_x mm	$I_{xy} = A (d_x)(d_y)$ mm^4 (10^{-6})
1	3600	35.7	-14.3	-1.84
2	2000	-64.3	25.7	-3.31
Total	5600	---	----	-5.15



Example of Mohr's Circle for Moment of Inertia

- On a sheet of graph paper, develop a scaled plot of Mohr's circle as follows.
- Plot point 'B' => (22.64, -5.15)
- Plot point 'A' => (3.85, 5.15)
- Draw a diameter between points A and B and construct the circle.
- Measure angle 2θ to determine the rotation of the principle axis
- From the scaled plot, read the values of I_{min} and I_{max}.
- All calculated and plotted values should be:
  - $I_x = 22.64 \times 10^6$ mm^4
  - $I_y = 3.85 \times 10^6$ mm^4
  - $I_{xy} = -5.15 \times 10^6$ mm^4
  - $I_{x'} = 23.95 \times 10^6$ mm^4
  - $I_{y'} = 2.53 \times 10^6$ mm^4
  - $\theta = 14.36^o$